# Kinematic formulation of rotational gas flow

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It is shown that for the steady isoenergetic rotational flow of an ideal gas, both the specific enthalpy and the speed of sound can be expressed as functions of the velocity. As a result, it is possible to formulate the equations of motion so that the velocity is the only dependent variable. For a gas whose enthalpy and sound speed are functionally related, the results are a generalization of those for a perfect gas. If the enthalpy and sound speed are independent variables, the new formulation leads to a single vector equation whose solution completely determines the flow.

## 1. Introduction

The steady flow of an ideal gas is governed by equations expressing the conservation of mass, momentum, and energy, supplemented by an appropriate equation of state. The dependent variables that are introduced can be classed into two groups. One consists of the kinematic variables, which are the components of the vector velocity. The other group contains the thermodynamic variables such as the pressure, density, entropy, etc. If the flow is irrotational, all these variables are expressible in terms of one new variable, the velocity potential. The problem is then reduced to the study of a single non-linear partial differential equation in one unknown. This reduction is in general not possible for a rotational flow. In that case, one may still be able to eliminate the thermodynamic variables, so that the equations contain only the vector velocity as an unknown. Such equations will be said to constitute the kinematic formulation of the flow. Although this new formulation may not make it easier to solve a given problem, it can be used to study flows by means of inverse methods.

The kinematic formulation of rotational flow was first obtained by Crocco (1937) for the special case of isoenergetic flow of a perfect gas in the absence of external forces. The results were extended by Prim (1952) to non-isoenergetic (but still adiabatic) flows of a more general gas obeying the power-law equation of state

$$\rho = p^k S(s), \tag{1.1}$$

where $\rho$, $p$, and $s$ are the density, pressure, and specific entropy. $S(s)$ is an arbitrary function of $s$ and $k$ is a constant. Prim also showed that all the thermodynamic variables except the pressure can be eliminated for an even more general gas obeying the product equation of state

$$\rho = P(p)\, S(s), \tag{1.2}$$

where $P(p)$ is an arbitrary function of $p$. These results have also been extended to diabatic flows (Gustafson & Krzywoblocki 1957).

For any real gas, there are temperature ranges in which internal modes of energy are excited. As long as the excitation process is fast enough to maintain local thermodynamic equilibrium in the flow, the gas can still be treated as an ideal gas, but with a more involved equation of state. For example, the excitation of rotational and vibrational modes of a polyatomic gas (without dissociation) can be represented by the equation of state of a thermally perfect gas. (This is a gas that still obeys the law $p = \rho RT$ with $R$ constant, but whose specific heats are functions of temperature.) A gas that is partially dissociated or ionized will in general have no simple equation of state. All the work on the kinematic formulation of rotational flow referred to above does not apply to any of these cases; even the thermally perfect gas law satisfies neither (1.1) nor (1.2). The question naturally arises as to whether a kinematic formulation can be constructed to include these cases. Current interest in this question is motivated by the prevalence of very high speed flows that give rise to high temperatures in strongly rotational stagnation regions.

The purpose of this paper is to show that at least for isoenergetic flows the question can be answered affirmatively. A surprising result of the investigation is that the flow of a gas obeying a perfectly arbitrary equation of state (such as a partially dissociated or ionized gas) possesses certain simple properties that are not present in a perfect or thermally perfect gas. Although external forces are neglected in the analysis, the results can easily be generalized to include them. In order to introduce the ideas in a simple manner, the case of incompressible flow will be discussed first.

## 2. Kinematic formulation of incompressible flow

The equations governing an incompressible flow are

$$\operatorname{div}\mathbf{V} = 0, \tag{2.1}$$

$$-(1/\rho)\operatorname{grad}p = \mathbf{V}.\operatorname{grad}\mathbf{V} = \operatorname{curl}\mathbf{V}\times\mathbf{V}+\tfrac{1}{2}\operatorname{grad}\mathbf{V}^2, \tag{2.2}$$

where $\mathbf{V}$ is the vector velocity. From (2.2) it follows that

$$\mathbf{V}.\operatorname{grad}(p+\tfrac{1}{2}\rho\mathbf{V}^2) = 0. \tag{2.3}$$

If all the streamlines pass through a region where $p+\frac{1}{2}\rho\mathbf{V}^2$ is constant (such as uniform flow), it follows from (2.3) that

$$p+\tfrac{1}{2}\rho\mathbf{V}^2 = p_0, \tag{2.4}$$

where $p_0$ is the stagnation pressure, which is a constant for the whole flow. If the gradient of (2.4) is substituted into (2.2), one obtains†

$$\operatorname{curl}\mathbf{V} = 0, \tag{2.5}$$

i.e. the flow is irrotational. Equations (2.1) and (2.5) constitute the kinematic formulation of an irrotational incompressible flow. By introducing a velocity

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† Equations (2.5) and (3.8) should actually be replaced by the less stringent condition for a Beltrami flow, namely $\mathbf{V}\times\operatorname{curl}\mathbf{V} = 0$. This does not affect the argument for a kinematic formulation. In most problems of interest, such as planar or axisymmetric flow, a Beltrami flow is also irrotational.

potential, determination of the velocity field is reduced to the solution of Laplace's equation. Once the velocity is known, the pressure is immediately obtained from (2.4).

In the general case where (2.4) and (2.5) do not hold, the pressure can still be eliminated by taking the curl of (2.2), resulting in

$$\operatorname{curl}(\mathbf{V}\times\operatorname{curl}\mathbf{V}) = 0. \tag{2.6}$$

Equations (2.1) and (2.6) constitute the kinematic formulation of a rotational incompressible flow. Equation (2.6) can be considered a compatibility relation which picks out those solutions of the continuity equation that are also dynamically possible. Once the velocity field is determined, the pressure can be obtained by integration from (2.2).

In comparing the formulations of rotational and irrotational flows, one notes that the former is more involved in two respects. The equations describing the velocity field are of higher order, and also non-linear. In addition, determination of the pressure requires an integration, instead of a simple arithmetical operation.

## 3. Kinematic formulation of compressible flow

### 3.1. *Irrotational flow*

The equations governing a compressible adiabatic flow are

$$\operatorname{div}(\rho\mathbf{V}) = 0, \tag{3.1}$$

$$-(1/\rho)\operatorname{grad}p = \mathbf{V}.\operatorname{grad}\mathbf{V} = \operatorname{curl}\mathbf{V}\times\mathbf{V}+\tfrac{1}{2}\operatorname{grad}\mathbf{V}^2, \tag{3.2}$$

$$\mathbf{V}.\operatorname{grad}s = 0. \tag{3.3}$$

Using the thermodynamic relation

$$\rho = \left(\frac{\partial p}{\partial h}\right)_s, \tag{3.4}$$

where $h$ is the specific enthalpy, (3.2) and (3.3) can be combined to yield

$$\mathbf{V}.\operatorname{grad}(h+\tfrac{1}{2}\mathbf{V}^2) = 0. \tag{3.5}$$

In the following analysis, it will be assumed that all the streamlines pass through a region where both $s$ and $h+\frac{1}{2}\mathbf{V}^2$ are constant (such as uniform flow). If the fact that the quantity $h+\frac{1}{2}\mathbf{V}^2$ is unchanged at a shock discontinuity is used, it follows from (3.5) that

$$h+\tfrac{1}{2}\mathbf{V}^2 = h_0, \tag{3.6}$$

where $h_0$ is the stagnation enthalpy, which is a constant for the whole flow. Equation (3.6) is the condition for an isoenergetic flow.

In the absence of shock waves, (3.3) leads to the additional condition

$$s = s_0, \tag{3.7}$$

i.e. the flow is isentropic. When (3.2), (3.4), and (3.6) are used, it immediately follows that

$$\operatorname{curl}\mathbf{V} = 0. \tag{3.8}$$

Therefore an isentropic isoenergetic flow is also irrotational. In order to complete the kinematic formulation of such a flow, it is necessary to eliminate the density from the continuity equation. Since the entropy is given, the density and pressure can be expressed as functions of the enthalpy for a given gas. In other words, the functions

$$\rho = g_1(h; s_0), \tag{3.9}$$

and

$$p = g_2(h; s_0) \tag{3.10}$$

are known once the entropy and gas are specified. Substitution of (3.6) and (3.9) into (3.1) yields

$$\operatorname{div}\left[g_1(h_0 - \tfrac{1}{2}V^2; s_0)\,\mathbf{V}\right] = 0. \tag{3.11}$$

Equations (3.8) and (3.11) constitute the kinematic formulation of the irrotational flow of an arbitrary gas. The introduction of a velocity potential results in a non-linear partial differential equation in one unknown. With the velocity determined, the pressure is then obtained from (3.10).

### 3.2. *General considerations for rotational flow*

It was seen above that the kinematic formulation of a compressible irrotational flow differs from the incompressible case only in that one of the equations is non-linear. The essential feature that the problem can be solved completely using a kinematic description has been preserved. In each case, the knowledge of the velocity field is all that is required, since the thermodynamic variables are then found by simple substitution.

If one now turns to the general case where (3.7) and (3.8) do not hold, one finds the situation more complicated. Equation (3.6) expresses the enthalpy as a function of the velocity, but the variables one must eliminate from (3.1) and (3.2) are the density and the pressure. The elimination of the pressure by taking the curl of the momentum equation as it stands is not possible. It would therefore be desirable to relate some other thermodynamic variable to the velocity.

Fortunately, such a relationship does exist. Equation (3.1) can be written

$$\mathbf{V}.\operatorname{grad}\rho = -\rho \operatorname{div}\mathbf{V}, \tag{3.12}$$

while from (3.2) it follows that

$$\mathbf{V}.\operatorname{grad}p = -\tfrac{1}{2}\rho\,\mathbf{V}.\operatorname{grad}V^2. \tag{3.13}$$

Using (3.3), one then obtains

$$\left(\frac{\partial p}{\partial \rho}\right)_s = \frac{\mathbf{V}.\operatorname{grad}V^2}{2\operatorname{div}\mathbf{V}}. \tag{3.14}$$

Here $(\partial p/\partial \rho)_s$ is, of course, equal to the square of the speed of sound in the gas. The significant results for the steady, compressible adiabatic flow of an ideal gas may therefore be summarized as follows.

*A knowledge of the kinematics of the flow (i.e. the velocity field) also determines the speed of sound at every point. If the flow is isoenergetic, it also determines the specific enthalpy at every point.*

The second result is well known, since it follows immediately from the definition of isoenergetic flow. However, the first result, which comes from (3.14), has not been fully appreciated. It is this equation that actually provides the key for the kinematic formulation of the flow of an arbitrary gas. In order to proceed further, one must distinguish between two types of gas laws—one in which $h$ and $(\partial p/\partial \rho)_s$ are functionally dependent, and the other in which they are independent variables. Each will lead to a different kinematic formulation. Of course, there exist flows in which both cases are found in different regions of the flow. In such an event, a different formulation must be used in the two regions. These two cases will be examined in the following sections.

### 3.3 *Rotational flow:* $(\partial p/\partial \rho)_s = f(h)$

In this section, it will be assumed that the speed of sound is functionally related to the specific enthalpy; i.e.

$$\left(\frac{\partial p}{\partial \rho}\right)_s = f(h). \tag{3.15}$$

With the aid of (3.4) it can be easily shown that the density must satisfy the relation

$$\rho = H(h)\,S(s), \tag{3.16}$$

where

$$H(h) = \exp\left[\int \frac{dh}{f(h)}\right]. \tag{3.17}$$

When (3.3), (3.6) and (3.16) are used, (3.1) may be rewritten as

$$\operatorname{div}\left[H(h_0 - \tfrac{1}{2}\mathbf{V}^2)\,\mathbf{V}\right] = 0. \tag{3.18}$$

The pressure law is obtained by integrating (3.4), using (3.16). Its general form is

$$p = A(h)\,S(s) + c(s), \tag{3.19}$$

where

$$A(h) = \int H(h)\,dh, \tag{3.20}$$

and $c(s)$ is an arbitrary function of $s$ which is not proportional to $S(s)$. In order to determine $c(s)$, one can use the boundary condition that

$$p(\rho = 0) = 0. \tag{3.21}$$

This relation follows from the first and second laws of thermodynamics, written in the form

$$T\,ds = dh - \frac{1}{\rho}\,dp, \tag{3.22}$$

where $T$ is the absolute temperature. As long as $T$ remains bounded when $\rho \to 0$, (3.22) requires that $dp \to 0$ also. Therefore $p = \text{const.}$ when $\rho = 0$, and since in this formulation, the pressure is defined only within an additive constant, the constant may be chosen equal to zero. An alternative proof depends on the relation between the specific enthalpy and the specific internal energy $u$, namely

$$h = u + p/\rho. \tag{3.23}$$

Equation (3.21) then follows immediately if one imposes the condition that the enthalpy and internal energy remain bounded as $\rho \to 0$.

If the range of validity of (3.15) includes the value of $h = a$ for which $H(a) = 0$, it follows from (3.16), (3.19), and (3.21) that $c(s) = 0$ and $A(a) = 0$. On the other hand, if (3.15) does not hold for $h = a$, then the laws of thermodynamics alone cannot determine $c(s)$. In order to eliminate the pressure from the momentum equation, it will be necessary to set

$$c(s) = 0. \tag{3.24}$$

Thus, if (3.15) is not valid at $h = a$, then (3.24) must be considered an additional assumption for the gas law.

Equations (3.16), (3.19), and (3.24) can be combined to yield

$$\rho = pB(h), \tag{3.25}$$

where

$$B(h) = \frac{H(h)}{A(h)} = \frac{\exp\left[\int \frac{dh}{f(h)}\right]}{\int \exp\left[\int \frac{dh}{f(h)}\right] dh}. \tag{3.26}$$

Substitution of (3.6) and (3.25) into (3.2) results in

$$B(h_0 - \tfrac{1}{2}\mathbf{V}^2)\,\mathbf{V}\,.\,\mathrm{grad}\,\mathbf{V} + \mathrm{grad}\,(\ln p) = 0. \tag{3.27}$$

The curl of (3.27) then gives the final form

$$\mathrm{curl}\,[B(h_0 - \tfrac{1}{2}\mathbf{V}^2)\,\mathbf{V} \times \mathrm{curl}\,\mathbf{V}] = 0. \tag{3.28}$$

Equations (3.18) and (3.28) constitute the kinematic formulation of the rotational flow of a gas that obeys (3.15) (and is also subject to condition (3.24), if necessary). If $\mathbf{V}^2 \ll 2h_0$, the equations reduce to (2.1) and (2.6), which are the corresponding expressions for an incompressible flow. It is seen that the compressible formulation differs chiefly in that the continuity equation is now also non-linear. With the velocity known, the pressure is obtained by integrating (3.27). The enthalpy, density, and entropy are then found by substitution from (3.6), (3.25), and (3.16). The temperature can be evaluated by using the following formula, derived from (3.19), (3.22), and (3.24):

$$p = -\rho T S(s)/S'(s). \tag{3.29}$$

A special case of interest is the thermally perfect gas law, $p = \rho RT$. From (3.25) and (3.29), it follows that

$$S(s) = \exp\,(-s/R); \quad B(h) = \frac{1}{RT}. \tag{3.30}$$

If the gas is also calorically perfect, it can be readily shown that

$$f(h) = (\gamma - 1)\,h, \tag{3.31}$$

where $\gamma$ is the ratio of specific heats. Equations (3.18), (3.27), and (3.28) then become

$$\mathrm{div}\,[(h_0 - \tfrac{1}{2}\mathbf{V}^2)^{1/(\gamma-1)}\,\mathbf{V}] = 0, \tag{3.32}$$

$$\mathbf{V}\,.\,\mathrm{grad}\,\mathbf{V} + \frac{\gamma - 1}{\gamma}\,(h_0 - \tfrac{1}{2}\mathbf{V}^2)\,\mathrm{grad}\,(\ln p) = 0, \tag{3.33}$$

$$\mathrm{curl}\left[\frac{\mathbf{V} \times \mathrm{curl}\,\mathbf{V}}{(h_0 - \tfrac{1}{2}\mathbf{V}^2)}\right] = 0. \tag{3.34}$$

This set of equations for a perfect gas was first derived by Crocco (1937).

### 3.4. *Rotational flow: $(\partial p/\partial \rho)_s$ and $h$ are independent variables*

The flow of a gas whose sound speed and enthalpy are independent variables will now be considered. Once such a gas is specified, the density and pressure are in principle known functions of these two variables; i.e.

$$\rho = F_1\left[h, \left(\frac{\partial p}{\partial \rho}\right)_s\right], \tag{3.35}$$

$$p = F_2\left[h, \left(\frac{\partial p}{\partial \rho}\right)_s\right]. \tag{3.36}$$

If (3.6), (3.14), (3.35), and (3.36) are substituted into (3.2), one obtains

$$F_1\left[(h_0-\tfrac{1}{2}\mathbf{V}^2), \frac{\mathbf{V}.\operatorname{grad}\mathbf{V}^2}{2\operatorname{div}\mathbf{V}}\right]\mathbf{V}.\operatorname{grad}\mathbf{V}+\operatorname{grad}F_2\left[(h_0-\tfrac{1}{2}\mathbf{V}^2), \frac{\mathbf{V}.\operatorname{grad}\mathbf{V}^2}{2\operatorname{div}\mathbf{V}}\right] = 0. \tag{3.37}$$

Equation (3.37) represents the kinematic formulation of the rotational flow of a gas whose sound speed and enthalpy are independent variables. Since (3.14), which is a combination of the continuity and momentum equations, was used in deriving (3.37), no additional equation is required. A velocity field which is a solution of (3.37) satisfies all the conservation laws governing the flow. The thermodynamic state of the gas is then obtained directly from (3.6), (3.14), (3.35), and (3.36). It is important to note that no additional problem needs to be solved to determine the dynamical variables. This is in contrast with the situation that exists not only for the flow of a gas in which $(\partial p/\partial \rho)_s = f(h)$, but even for an incompressible flow. In both these cases, the velocity field given by the kinematic equations does not automatically determine the dynamical variables, but a further vectorial integration is required. In this respect, the rotational flow of a perfectly arbitrary gas possesses a simplicity which is not present in a more specialized fluid. This is due to the fact that the kinematics of the flow brings with it knowledge concerning two thermodynamic variables. For a general gas, the knowledge is sufficient to determine the complete dynamics of the problem. But a gas that in this sense is degenerate already possesses some of this knowledge and therefore does not extract from the kinematics all the information that it needs. Consequently, the kinematic formulation of the flow of such a gas requires one more equation, and the determination of the dynamics requires an additional integration. Inasmuch as it completely uncouples the kinematics and the dynamics, the rotational flow of an arbitrary gas may even be said to resemble an irrotational flow.

For any given gas, (3.37) can be specialized further, since the functions $F_1$ and $F_2$ are not independent. They can be indirectly related through the equation

$$\left(\frac{\partial p}{\partial \rho}\right)_s^{-1} = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial h}\right)_p + \left(\frac{\partial \rho}{\partial p}\right)_h, \tag{3.38}$$

which follows directly from (3.4). To illustrate, two simple types of gas laws will be considered. The first is

$$\rho = bp + G(h), \tag{3.39}$$

where $b$ is a constant and $G(h)$ is an arbitrary function of $h$. Substituting (3.39) into (3.38), and solving for $p$, one obtains the expression for $F_2$:

$$F_2\left[h, \left(\frac{\partial p}{\partial \rho}\right)_s\right] = -\frac{1}{b}\left[\frac{G'(h)}{b-\left(\frac{\partial p}{\partial \rho}\right)_s^{-1}} + G(h)\right], \tag{3.40}$$

while $F_1$ becomes

$$F_1\left[h, \left(\frac{\partial p}{\partial \rho}\right)_s\right] = bF_2\left[h, \left(\frac{\partial p}{\partial \rho}\right)_s\right] + G(h) = \frac{-G'(h)}{b-\left(\frac{\partial p}{\partial \rho}\right)_s^{-1}}. \tag{3.41}$$

As a second example, let $w(p)$ be a function whose inverse can be written explicitly. Define

$$W(p) = \int w(p)\,dp. \tag{3.42}$$

Then if

$$\rho = \frac{W(p)}{G(h)}, \tag{3.43}$$

it follows from (3.38) that

$$w(p) = \frac{G(h)}{\left(\frac{\partial p}{\partial \rho}\right)_s} + G'(h), \tag{3.44}$$

and consequently

$$F_2\left[h, \left(\frac{\partial p}{\partial \rho}\right)_s\right] = w^{-1}\left[\frac{G(h)}{\left(\frac{\partial p}{\partial \rho}\right)_s} + G'(h)\right], \tag{3·45}$$

while

$$F_1\left[h, \left(\frac{\partial p}{\partial \rho}\right)_s\right] = \frac{W(F_2)}{G(h)}. \tag{3.46}$$

A special case of (3.43) is the relation

$$h = h_1 + \nu\frac{p}{\rho^n}, \tag{3.47}$$

where $h_1$, $\nu$ and $n$ are constants. Equation (3.47) has been suggested (Wohlwill 1957) as giving a good approximation to the equation of state of partially dissociated air in the subsonic region behind a detached shock wave. If (1.2) is integrated using (3.4) (and the arbitrary function of $s$ that is introduced is set equal to zero), one obtains

$$\rho h = P(p)\int\frac{dp}{P(p)}. \tag{3.48}$$

Thus, (1.2) is also seen to be a special case of (3.43).

## 4. Discussion of results

The main results of this investigation are embodied in (3.18), (3.28), and (3.37). Taken together, these equations give a kinematic representation of the rotational flow of any gas (except one obeying (3.19) with $c(s) \neq 0$). The complete formulation of a flow problem requires the specification of boundary conditions. These pose no difficulty since they can also be expressed kinematically, as is shown below.

It has been implicitly assumed in this analysis that the stagnation enthalpy $h_0$ is a known quantity. This is certainly the case if one of the conditions of the problem is uniform flow at infinity. At any rate, the specification of $h_0$ is one of the boundary conditions. There are two types of boundary conditions involving geometric surfaces. One is a solid boundary, at which the normal component of velocity must be zero. The other is a surface of discontinuity, such as a shock wave. Here the tangential momentum equation leads to the condition that there is no jump in tangential velocity. The continuity and normal momentum equations combined lead to a relation involving the normal velocity and the quantity

$p/\rho$ on each side of the shock wave. But from (3.6), (3.14), (3.25), (3.35), and (3.36), it follows that no matter which type of gas exists on either side, $p/\rho$ can be expressed as a function of the velocity. Therefore, the shock boundary conditions can also be expressed in kinematic form.

A few remarks should be made concerning the application of the two types of gas laws to an actual gas. At pressures normally encountered in flow problems, any gas is well approximated by an at least thermally perfect gas law up to the temperatures where dissociation or ionization become non-negligible. The formulation of § 3.3 applies in this case. Another region of applicability is for a polyatomic gas which dissociates completely before it starts to ionize. In this temperature region, such a gas becomes monatomic and again behaves like a perfect gas. Both are examples of gases that obey (3.15) and also (3.24). On the basis of the behaviour of known gases, it seems doubtful that (3.15) can hold without at the same time (3.24) being true.

The general gas law described in § 3.4 should apply to all other gases. This includes a gas at such high pressures that cohesive forces and the finite volume of molecules force departures from the perfect gas law. A wider range of applicability is for a gas that is partially dissociated or ionized. A difficulty in practice is that explicit forms of (3.35), and (3.36) will usually not exist. For limited regions of the flow, special forms of the gas law such as (3.39) or (3.43) which do give rise to explicit expressions for $F_1$ and $F_2$ may be found useful.

It might be instructive as an illustration to consider the uniform flow of a polyatomic gas past a blunt-nosed body over a wide range of speeds. At very low speeds, the flow is incompressible and irrotational and is governed by equations (2.2) and (2.5). The problem is linear, and the kinematics and dynamics are completely uncoupled. As the speed increases, and compressibility can no longer be neglected, (2.2) has to be replaced by (3.11). This introduces a non-linearity, but the dynamics and kinematics are still uncoupled. As the critical Mach number is exceeded, shocks begin to form, and behind them the flow becomes rotational. When the uniform flow becomes supersonic, a detached shock wave forms ahead of the body, and the entire flow between the body and the shock wave is rotational. The gas initially will behave like a perfect gas, and will be described by (3.32) and (3.34). Not only have both equations become non-linear, but the integration of (3.33) is now required to determine the pressure, once the velocity is known. At still higher speeds the temperature in the nose region becomes high enough to excite vibration. The gas departs from a perfect gas, and the more general (3.18) and (3.28) must be used. As the uniform flow becomes hypersonic the gas in the nose region starts to dissociate. The flow is now described completely by (3.37), and the dynamics and kinematics are uncoupled again. If the gas dissociates completely before it begins to ionize, increasing the velocity even further will result in a flow near the nose that is again described by (3.32) and (3.34).

The above example shows that the various types of rotational gas flows considered in this paper can easily occur in practice. There is a question whether the kinematic formulation of such flows can actually be useful in their solution. To answer this, one must first consider simpler problems. These include the simplification of the equations for two-dimensional (planar or axisymmetric) flows, and the search for solutions that have simple kinematic properties by using inverse methods. Much of this work has already been done for the case of a perfect gas law (Prim 1952). The extension of some of these results to the more general types of gas laws considered here will be reported in a future publication.

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